Synchronism *vs* Asynchronism in Boolean automata networks

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Abstract. We show that synchronism can significantly impact on network behaviours, in particular by filtering unstable attractors induced by a constraint of asynchronism. We investigate and classify the different possible impacts that an addition of synchronism may have on the behaviour of a Boolean automata network. We show how these relate to some strong specific structural properties, thus supporting the idea that for most networks, synchronism only shortcuts asynchronous trajectories. We end with a discussion on the close relation that apparently exists between sensitivity to synchronism and non-monotony.

Keywords: Automata network, synchronism, asynchronism, attractor, updating mode, elementary transition, atomic transition.

Introduction

In works involving automata networks, synchronism has often either been considered as a founding hypothesis, as in [10] and the many studies that followed in its lead, or, on the contrary, in lines with [18], it has been disregarded altogether to the benefit of pure asynchrony. In some applied contexts, *theoretical synchronism* is sometimes understood as *simultaneity* although this restrictive interpretation relies on a formalisation of duration which conflicts with the discrete nature of automata networks. More simply, the synchronous occurrence of two changes in a network can be regarded as occurrences that are close enough in time to disallow any other significant event in between them. This naturally defines a much more general notion of time flow that is strongly *relative* to the set of events underwent by the network. And thus it yields substantial representational capacity to the notion of synchronism, justifying the attention that we propose to give to it in this paper.

Comparisons have been made between different kinds of ways of updating automata states, involving variable degrees of synchronism in both probabilistic [2,6,7,12,17] (with cellular automata) and deterministic frameworks [1,4,5,9,15,16]. In particular, for the algorithmic purpose of finding the shortest path to a stable configuration, Robert[16] compared Boolean automata network behaviours under the parallel and sequential update schedules. In this context, he noted three

"frequent (but not systematic) phenomena" that can be observed through the effect of parallelisation: the "*bursting*", the "*aggregation*" and the "*implosion*" of attraction basins. Here, we focus on attractors, both stable and unstable. And considering more generally state transition systems rather than just deterministic dynamical systems, we propose to investigate synchronism *per se*, and analyse its *input* to the design of Boolean automata network behaviours. More precisely, we propose to consider *asynchronous* transition graphs representing the set of all punctual and atomic events of a network and we propose to explore the consequences of adding to it a synchronous transition, representing a new possibility to perform a punctual but non-atomic change. We propose to identify the cases where such an addition of synchronism changes substantially a network's possible asymptotic behaviours or its evolutions towards them. Thus, we are looking for networks for which synchronism does not just shortcut asynchronous trajectories but rather also adds some new ones that can not be *mimicked* asynchronously. This leads to classify the possible impacts of *non-sequentialisable transitions* and then the sensitivity of networks to synchronism.

Preliminaries

Notations – By default, $\mathbf{V} = \{0, \ldots, n-1\}$ denotes a set of $n \in \mathbb{N}$ automata numbered from 0 to $n-1$. We let $\mathbb{B} = \{0, 1\}$. Any $x \in \mathbb{B}^n$ is called a **configuration** and its component $x_i \in \mathbb{B}$ is regarded as the **state** of automaton $i \in \mathbf{V}$. In this paper, special attention is paid to switches of automata states starting in a given configuration. For this reason, we introduce the following notations:

$$\forall x = x_0 \ldots x_{n-1} \in \mathbb{B}^n, \forall i \in \mathbf{V},\ \overline{x}^{\,i} = x_0 \ldots x_{i-1} \neg x_i\, x_{i+1} \ldots x_{n-1}$$
$$\text{and } \forall \mathrm{W} = \mathrm{W}' \uplus \{ i \} \subseteq \mathbf{V},\ \overline{x}^{\,\mathrm{W}} = \overline{(\overline{x}^{\,i})}^{\,\mathrm{W}'} = \overline{(\overline{x}^{\,\mathrm{W}'})}^{\,i}.$$

Also, to compare two configurations $x, y \in \mathbb{B}^n$, we use: $\mathrm{D}(x, y) = \{ i \in \mathbf{V};\ x_i \neq y_i \}$ and the Hamming distance $\mathrm{d}(x, y) = |\,\mathrm{D}(x, y)|$. Finally, to switch values from $\mathbb{B}$ to $\{-1, 1\}$, we let $\mathbf{s} : b \in \mathbb{B} \mapsto b - \neg b \in \{-1, 1\}$.

Networks – A **Boolean automata network** (BAN) of size n is comprised of n interacting automata. Formally, it is a set $\mathcal{N} = \{ \mathrm{f}_i\, ;\, i \in \mathbf{V} \}$ of n Boolean functions specifying "how the net of automata works". Function $\mathrm{f}_i : \mathbb{B}^n \to \mathbb{B}$ specifies the behaviour of automaton $i \in \mathbf{V}$ in any configuration $x \in \mathbb{B}^n$. It is called the **transition function** of i. We focus on functions that are *locally monotone* w.r.t. all their components, *i.e.* $\forall i \in \mathbf{V}, \forall j \in \mathbf{V}$ we assume:

$$\text{that either } \forall x \in \mathbb{B}^n,\ \mathbf{s}(x_j) \cdot (\mathrm{f}_i(x) - \mathrm{f}_i(\overline{x}^{\,j})) \geq 0 \tag{1}$$
$$\text{or } \forall x \in \mathbb{B}^n,\ \mathbf{s}(x_j) \cdot (\mathrm{f}_i(x) - \mathrm{f}_i(\overline{x}^{\,j})) \leq 0. \tag{2}$$

At the end of this paper, non-monotony is discussed. Until then, we assume all BANs to be **monotone**, that is, to involve transition functions that are locally monotone w.r.t. all their components.

Network structures – The **structure** of $\mathcal{N}$ is the digraph $\mathbf{G} = (\mathbf{V}, \mathbf{A})$ whose node set is $\mathbf{V}$ (thus, automata are also called nodes) and whose arc set is: $\mathbf{A} = \{ (j,i) \in \mathbf{V}^2;\ \exists x \in \mathbb{B}^n,\ \mathrm{f}_i(x) \neq \mathrm{f}_i(\overline{x}^{\,j}) \}$. $\mathbf{V}^-(i)$ denotes the in-neighbourhood of $i \in \mathbf{V}$ in $\mathbf{G}$. The local monotony of transition functions allows us to **sign** the arcs of $\mathbf{G}$. $\forall (j,i) \in \mathbf{A}$ and $\forall x \in \mathbb{B}^n$ s.t. $\mathrm{f}_i(x) \neq \mathrm{f}_i(\overline{x}^{\,j})$, we let $\mathrm{sign}(j,i) = \mathbf{s}(x_j) \cdot (\mathrm{f}_i(x) - \mathrm{f}_i(\overline{x}^{\,j})) = \mathbf{s}(x_j) \cdot \mathbf{s}(\mathrm{f}_i(x))$ which equals $+1$ if (1) is satisfied and -1 if (2) is. We let $\mathrm{sign}(j,i) = 0$ when $(j,i) \notin \mathbf{A}$. Naturally, we define the sign of a path or cycle in $\mathbf{G}$ as the product of the signs of the arcs it involves. Thus, a positive path globally transmits "information" directly whereas a negative one transmits its negation.

Instabilities and frustrations – For every $x \in \mathbb{B}^n$, we define the set: $\mathbf{U}(x) = \{ i \in \mathbf{V};\ \mathrm{f}_i(x) \neq x_i \}$. Automata in $\mathbf{U}(x)$ are said to be **unstable** (or "calling for a change or updating"[13]) in x and those in $\overline{\mathbf{U}}(x) = \mathbf{V} \setminus \mathbf{U}(x)$ are said to be **stable** in x. Informally, the number $\mathrm{u}(x) = |\mathbf{U}(x)|$ of instabilities in x can be understood as the *velocity* or *momentum* of $\mathcal{N}$ in x. Configurations x such that $\mathrm{u}(x) = 0$ are called **stable**. Our first lemma which will be very useful in the sequel, relates instabilities to BAN structures. Its proof is simple so we skip it.

Lemma 1 (loops). $\forall i \in \mathbf{V}, \forall x \in \mathbb{B}^n,\ i \in \overline{\mathbf{U}}(x) \cap \overline{\mathbf{U}}(\overline{x}^{\,i}) \Rightarrow \mathrm{sign}(i,i) = +1$ *and* $i \in \mathbf{U}(x) \cap \mathbf{U}(\overline{x}^{\,i}) \Rightarrow \mathrm{sign}(i,i) = -1$.

$\forall x \in \mathbb{B}^n$, we introduce the set of **arcs that are frustrated[3,8,19,20] in** x:

$$\textsc{frus}(x) = \{ (j,i) \in \mathbf{A}\,;\ \mathbf{s}(x_j) \cdot \mathbf{s}(x_i) = -\mathrm{sign}(j,i) \} \tag{3}$$

Our second preliminary lemma states that adding frustrated arcs incoming an unstable automaton cannot make it stable. Again, we skip its proof which mainly relies on the local monotony of transition functions.

Lemma 2 (frustrations & instabilities). $\forall i \in \mathbf{V},\ \forall x, y \in \mathbb{B}^n$: $\big(i \in \mathbf{U}(x) \ \wedge\ \textsc{frus}(x) \cap \mathbf{V}^-(i) \subset \textsc{frus}(y) \cap \mathbf{V}^-(i) \big) \ \Rightarrow\ i \in \mathbf{U}(y)$.

Transitions and and transition graphs – An **elementary transition** of a BAN $\mathcal{N}$ represents an *effective*, *punctual* and possible change in $\mathcal{N}$. It is any couple of configurations $(x,y) \in \mathbb{B}^n \times \mathbb{B}^n$, noted $x \longrightarrow y$, which satisfies: $\emptyset \neq \mathrm{D}(x,y) \subseteq \mathbf{U}(x)$. The **size of an elementary transition** $x \longrightarrow y$ equals $\mathrm{d}(x,y)$. Digraph $(\mathbb{B}^n, \{ x \longrightarrow y;\ x, y \in \mathbb{B}^n \})$ is called the **elementary transition graph** (ETG) of $\mathcal{N}$. It represents all punctual/elementary events that $\mathcal{N}$ can undergo. There are two main types of elementary transitions $x \longrightarrow y$. Those of size $\mathrm{d}(x,y) > 1$ are called **non-atomic** or **synchronous** and are written $x \longrightarrow\!\!\blacktriangleright y$. Those of size $\mathrm{d}(x,y) = 1$ are called **asynchronous** or **atomic** and are written $x \longrightarrow\!\!\triangleright y$ (they are s.t. $\exists i \in \mathbf{V}$, $y = \overline{x}^{\,i}$). Digraph $\mathcal{T}_{\mathrm{a}} = (\mathbb{B}^n, \{ x \longrightarrow\!\!\triangleright y;\ x, y \in \mathbb{B}^n \})$ is called the **asynchronous transition graph** or ATG. It represents only those events that $\mathcal{N}$ can undergo which involve only one local automaton state change. The transitive closure of $\longrightarrow$ (resp. $\longrightarrow\!\!\triangleright$) is denoted by $\longrightarrow\!\!\!\!\rightarrow$ (resp. $\longrightarrow\!\!\triangleright\!\!\triangleright$). **Derivations** are ordered lists of these non necessarily elementary transitions written $x^0 \longrightarrow\!\!\!\!\rightarrow x^1 \longrightarrow\!\!\!\!\rightarrow \ldots \longrightarrow\!\!\!\!\rightarrow x^{\ell-1} \longrightarrow\!\!\!\!\rightarrow x^\ell$ but in the sequel, we abuse language and also speak of a *derivation* $x \longrightarrow\!\!\!\!\rightarrow y$.

Non-sequentialisable transitions and critical cycles

A **cycle** of a BAN $\mathcal{N}$ is a sub-graph of its structure $\mathbf{G}$ corresponding to a closed directed walk, with possibly repeated nodes but no repeated edges. $\forall x \in \mathbb{B}^n$, we say that a cycle $\mathbf{C} = (\mathbf{V_C}, \mathbf{A_C})$ of $\mathcal{N}$ is **x-critical** if: $\mathbf{V_C} \subset \mathbf{U}(x) \;\wedge\; \mathbf{A_C} \subset \textsc{frus}(x)$. Note that for an isolated cycle, since $|\mathbf{V}^-(i)| = 1$, $\forall i \in \mathbf{V_C}$, a node is unstable if and only if its sole incoming arc is frustrated. A **critical cycle** of $\mathcal{N}$ is one that is x-critical for some $x \in \mathbb{B}^n$. All main results of this paper mention these types of cycles. This yields some importance to Proposition 1 below which derives from the following which, by (3), holds for any x-critical cycle $\mathbf{C} = (\mathbf{V_C}, \mathbf{A_C})$ of length ℓ and sign $\mathbf{s}$: $\prod_{(j,i)\in\mathbf{A_C}} -\mathrm{sign}(j,i) = (-1)^\ell \times \mathbf{s} = \prod_{(j,i)\in\mathbf{A_C}} \mathbf{s}(x_j)\cdot\mathbf{s}(x_i) = 1.$

Proposition 1. *A cycle that is critical is either positive with an even length or negative with an odd length.*

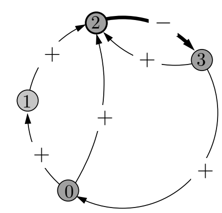


Fig. 1. Signed BAN structure whose Hamiltonian cycle $\mathbf{C} = (\mathbf{V_C}, \mathbf{A_C})$ is as in Proposition 1. If $\mathrm{f}_2 : x \mapsto x_3 \wedge (x_0 \vee x_1)$, then $\mathbf{C}$ cannot be critical because $2 \in \mathbf{U}(x)$ and $\mathbf{A_C} \subset \textsc{frus}(x)$ cannot be satisfied at once.

Let us emphasise that although a positive (resp. negative) cycle with an even (resp. odd) length is critical when it is isolated, when embedded in larger structures, it may loose this property (*cf.* Figure 1).

Now, the next result sets the backbone of the article: it shows how critical cycles are the main structural aspects of a BAN underlying its possibility to perform synchronous changes that cannot be mimicked asynchronously. First, let us say that $x \longrightarrow y$ is **sequentialisable** if it is asynchronous or if it can be broken into an derivation $x \twoheadrightarrow y$ involving smaller transitions. A synchronous transition $x \longrightarrow y$ which is not sequentialisable is called a **normal transition** and is rather written $x \Longrightarrow y$.

Proposition 2 (sequentialisable transitions and critical cycles). *Let $x \longrightarrow y$ be a synchronous transition of an arbitrary* BAN *$\mathcal{N}$. There is a derivation $x \longrightarrow \overline{x}^{\mathrm{D}_0} \longrightarrow \overline{x}^{\mathrm{D}_0 \uplus \mathrm{D}_1} \longrightarrow \ldots \longrightarrow \overline{x}^{\mathrm{D}_0 \uplus \mathrm{D}_1 \ldots \uplus \mathrm{D}_{m-1}} = y$ of $\mathcal{N}$ such that $\mathrm{D}(x,y) = \biguplus_{t<m} \mathrm{D}_t$ and $\forall t < m$, $|\,\mathrm{D}_t\,| > 1$ holds only if all automata of D_t belong to the same x-critical cycle.*

A crucial consequence of Proposition 2 is that any synchronous transition $x \longrightarrow y$ is sequentialisable as long as the automata in $\mathrm{D}(x,y)$ do not all belong the same x-critical cycle. And it is *totally* sequentialisable ($x \longrightarrow\!\!\rhd\!\!\rhd y$) if no subset of $\mathrm{D}(x,y)$ induces a x-critical cycle. Generally, Proposition 2 implies that for a BAN with no critical cycles of size $m \in \mathbb{N}$ or less, any synchronous change of $m' \leq m$ automata states can be totally sequentialised.

Proof. Consider the digraph $\mathbf{H} = (\mathrm{D}(x,y), \textsc{frus}(x))$ and let $\delta : \mathrm{D}(x,y) \to \{\,0, 1, \ldots, m{-}1\,\}$ be a topological ordering of the nodes of $\mathbf{H}$: $\forall j, i \in \mathrm{D}(x,y),\ (j,i) \in$

$\textsc{frus}(x) \;\Rightarrow\; \delta(i) \leq \delta(j)$ s.t. if j and i do not belong to the same cycle in $\mathbf{H}$ (and thus do not belong to the same x-critical cycle of $\mathbf{G}$), then $(j, i) \in \textsc{frus}(x) \;\Rightarrow\; \delta(i) < \delta(j)$. Now, let $\mathrm{D}_t = \{\, i \in \mathrm{D}(x, y),\ \delta(i) = t \,\}$ and $x(0) = x$. Based on Lemma 2, an induction on $t < m$ proves that $\forall t < m,\ x(t) \longrightarrow x(t{+}1) = \overline{x(t)}^{\mathrm{D}_t}$ is a transition of $\mathcal{N}$. □

The next lemma considers the case where the only critical cycles of $\mathcal{N}$ are Hamiltonian cycles.

Lemma 3. *Let $\mathcal{N}$ be a* BAN *whose critical cycles all have node set $\mathbf{V}$. Then, either $\mathcal{N}$ has a unique transition $x \Longrightarrow y$, or it has two $x \Longrightarrow y$ and $y \Longrightarrow x$. In the first case, every $i \in \overline{\mathbf{U}}(y)$ bears a positive loop $(i, i) \in \mathbf{A}$. In both cases the endpoints of these transitions can be reached by no asynchronous derivation.*

Proof. Suppose that $x \Longrightarrow y$ and $x' \Longrightarrow y'$ are two normal transitions. Using Proposition 2, if $x' \neq y$, then $\mathrm{W} = \mathrm{D}(x, x') \subsetneq \mathbf{U}(x) = \mathbf{V}$ and $\mathrm{D}(x', y) = \mathbf{V} \setminus \mathrm{W} \subsetneq \mathbf{U}(x') = \mathbf{V}$. In this case, $x \longrightarrow x' \longrightarrow y$ is a derivation of $\mathcal{N}$ involving smaller transitions than $x \longrightarrow y$, in contradiction with $x \longrightarrow y$ being normal. Thus, if $x \Longrightarrow y$ is not the unique normal transition of $\mathcal{N}$, then the only other one is $y \Longrightarrow x$. For any normal transition $z \Longrightarrow z' = \overline{z}^{\mathbf{V}}$, and $\forall i \in \mathbf{V},\ z \longrightarrow \overline{z'}^{i}$ is a transition of $\mathcal{N}$. By hypothesis and by Proposition 2, it is sequentialisable: $z \longrightarrow\!\!\triangleright\!\triangleright \overline{z'}^{i}$. Since $z \Longrightarrow z'$ is not however, this implies $i \in \overline{\mathbf{U}}(\overline{z'}^{i})$, $\forall i \in \mathbf{V}$. Thus, the endpoint of any normal transition of $\mathcal{N}$ can be reached by no asynchronous derivation. And since $\forall i \in \mathbf{V},\ i \in \overline{\mathbf{U}}(\overline{y}^{i})$, any $i \in \overline{\mathbf{U}}(y)$ is such that $\mathrm{sign}(i, i) = +1$ by Lemma 1. □

Impact of synchronous transitions

Let us introduce some new vocabulary to describe the transition graphs of $\mathcal{N}$. Stable configurations and terminal strongly connected components of these graphs are called **attractors** (abusing language because it may be that an attractor doesn't attract anything). Attractors that are not stable configurations are said to be unstable. Now, while presenting notations relative to the ATG (underscripted by 'a'), let us continue introducing terminology relative to any transition graph of $\mathcal{N}$, in particular its ETG. $\forall x \in \mathbb{B}^n$, we let $\mathcal{O}_{\mathrm{a}}(x) = \{\, y \in \mathbb{B}^n \,;\, x \longrightarrow\!\!\triangleright\!\triangleright y \,\}$ and $\mathcal{B}_{\mathrm{a}}(x) = \{\, y \,;\, y \longrightarrow\!\!\triangleright\!\triangleright x \,\}$. Also, we let $\mathcal{A}_{\mathrm{a}}(x)$ denote the set of attractors that x can reach in $\mathcal{T}_{\mathrm{a}}$. We say that a configuration x is **recurrent** when it belongs to an attractor and we denote this attractor by $[x]_{\mathrm{a}}$ (then, $\mathcal{A}_{\mathrm{a}}(x) = \{\, [x]_{\mathrm{a}} \,\}$). The basin of an attractor $[x]_{\mathrm{a}}$ is $\mathcal{B}_{\mathrm{a}}([x]_{\mathrm{a}}) = \mathcal{B}_{\mathrm{a}}(x) \setminus [x]_{\mathrm{a}}$. Non-recurrent configurations are called **transient**.

Let us consider an arbitrary synchronous transition $x \longrightarrow\!\!\blacktriangleright y$ of $\mathcal{N}$ and let $\mathcal{T}_{\mathrm{a}}{}' = (\mathbb{B}^n, \mathrm{T}_{\mathrm{a}} \cup \{\, (x, y) \,\})$ denote the transition graph obtained by adding this transition to the ATG $\mathcal{T}_{\mathrm{a}}$. We introduce notations $\mathcal{A}(z)$, $\mathcal{B}(z)$, $\mathcal{O}(z)$ and $[x]$ relative to $\mathcal{T}_{\mathrm{a}}{}'$ naturally as we did above for $\mathcal{T}_{\mathrm{a}}$. In the sequel, we say that an attractor A of $\mathcal{T}_{\mathrm{a}}$ is *destroyed* by $x \longrightarrow\!\!\blacktriangleright y$ if all its configurations are transient in $\mathcal{T}_{\mathrm{a}}{}'$. Generally, since $\forall z \in \mathcal{B}_{\mathrm{a}}(x) \cup \{\, x \,\},\ \mathcal{A}(z) = \mathcal{A}_{\mathrm{a}}(z) \cup \mathcal{A}_{\mathrm{a}}(y),$ the addition of $x \longrightarrow\!\!\blacktriangleright y$ to $\mathcal{T}_{\mathrm{a}}$ can

have several possible consequences on the asymptotic evolution of $\mathcal{N}$ starting in a configuration $z \in \mathcal{B}_a(x) \cup \{ x \}$. We list them now exhaustively.

1. We say that it has **no impact** when x is transient in $\mathcal{T}_a$ and $\mathcal{A}_a(y) \subset \mathcal{A}_a(x) = \mathcal{A}(x)$. In this case, $x \longrightarrow y$ 'only' adds to $\mathcal{T}_a$ some new derivations from x to the configurations of the orbit $\mathcal{O}_a(x)$ of x. It does not change the result of any network evolution. In particular, if $x \longrightarrow y$ is sequentialisable, then it shortcuts some derivations starting in x. But on the contrary, it can also deviate some derivations (when $\exists z \in \mathcal{O}_a(x) \cap \mathcal{O}_a(y)$ s.t. $y \twoheadrightarrow z$ is no shorter than $x \twoheadrightarrow z$).

Obviously, all synchronous transitions $x \longrightarrow y$ that *do* have an impact on the asymptotic evolution of $\mathcal{N}$ are normal.

2. We say that transition $x \boldsymbol{\Longrightarrow} y$ has little or **F-impact** (*cf.* Figure 2) if x is transient in $\mathcal{T}_a$ and $\mathcal{A}(x) = \mathcal{A}_a(x) \cup \mathcal{A}_a(y) \neq \mathcal{A}_a(x)$. Here, the addition of $x \boldsymbol{\Longrightarrow} y$ adds some new degrees of freedom to the asymptotic outcomes of the evolutions of $\mathcal{N}$ from any configuration $z \in \mathcal{B}_a(x) \cup \{ x \}$.Thus, it causes the growth of the basins $\mathcal{B}_a(A),\ A \in \mathcal{A}_a(y)$.

$$\forall x \in \mathbb{B}^n, \begin{cases} \mathrm{f}_0(x) = x_0 \wedge \neg x_1 \\ \mathrm{f}_1(x) = \neg x_0 \wedge x_1 \end{cases}$$

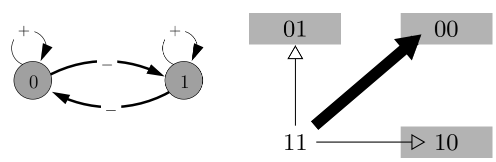


Fig. 2. Transition functions, structure and modified ATG $\mathcal{T}_a{}'$ of a BAN $\mathcal{N}$ whose normal transition 11 $\boldsymbol{\Longrightarrow}$ 00 has F-impact. This is a special case of F-impact induced by one critical, Hamiltonian cycle with no automata outside of it. Its impact, precisely, consists in making reachable an reachable attractor (*cf.* Lemmas 3 and 4).

Note that with addition of synchronous transitions that have no or F-impact, the set of recurrent configurations of $\mathcal{T}_a$ equals that of $\mathcal{T}_a{}'$.

3. We say that transition $x \boldsymbol{\Longrightarrow} y$ has **G-impact** (*cf.* the end of Example 1) on the asymptotic evolution of $\mathcal{N}$ when, in $\mathcal{T}_a$, x is recurrent, y is transient and $\mathcal{A}_a(y) = \mathcal{A}_a(x) = \{[x]_a\}$. In this case, y is connected to x and recurrent in $\mathcal{T}_a{}'$. The addition of $x \boldsymbol{\Longrightarrow} y$ to $\mathcal{T}_a$ causes attractor $[x]_a$ to absorb all derivations from y to $[x]_a$ and grow into $[x] = [y]$ without being destroyed.
4. We say that transition $x \boldsymbol{\Longrightarrow} y$ has **D-impact** (*cf.* Example 1) if x and y are both recurrent in $\mathcal{T}_a$ and $\mathcal{A}_a(y) \setminus \mathcal{A}_a(x) \neq \emptyset$. In this case, the addition of $x \boldsymbol{\Longrightarrow} y$ destroys the unstable attractor $[x]_a$ by emptying it into (the basins of) the attractors $A \in \mathcal{A}_a(y) \setminus \mathcal{A}_a(x)$.

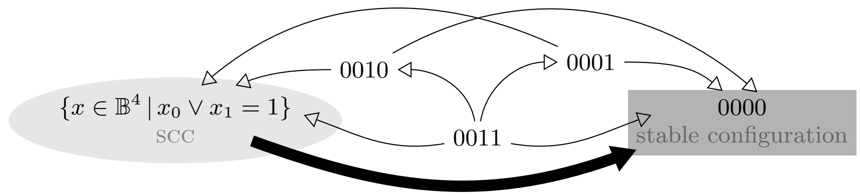


Fig. 3. Schematic representation of $\mathcal{T}_{\mathrm{a}}{}'$, the ATG of the BAN of Example 1 augmented with normal transition 1100 $\Longrightarrow$ 0000 which updates both automata 0 and 1 simultaneously. The shaded ellipse corresponds a strongly connected component which is terminal in $\mathcal{T}_{\mathrm{a}}$ but not in $\mathcal{T}_{\mathrm{a}}{}'$ nor the ETG with the added possibility of 1100 $\Longrightarrow$ 0000.

It can be checked that the four types of impact listed above are disjoint and cover all possible cases. It follows as a particular consequence that a unique normal transition is not enough to merge attractors. Let us emphasise that a configuration that is recurrent in the ATG can become transient with the addition of synchronism (if $\mathcal{N}$ has a normal transition with D-impact), in particular, it can become transient in the ETG. Conversely, synchronism can turn a transient configuration into a recurrent one (if $\mathcal{N}$ has a transition with G-impact). Synchronism can however not create new attractors from scratch. Indeed, if all configurations of a set $X \subset \mathbb{B}^n$ are transient in the ATG, then in the ETG as well as in its sub-graph the ATG, there necessarily exists a derivation outgoing X.

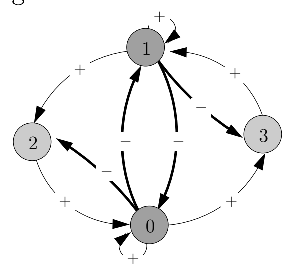
The addition of $x \Longrightarrow y$ to the ATG has no or little (*i.e.* F-) impact when x is transient in the ATG. To change the asymptotics of $\mathcal{N}$ (rather than just some of its evolutions towards it), $x \Longrightarrow y$ needs to have G- or D-impact. And for this, in the ATG, an unstable attractor $[x]_{\mathrm{a}}$ is needed. It can only be induced by a negative cycle in the structure $\mathbf{G}$ of $\mathcal{N}$ [14]. Further, considering Hamiltonian critical cycles again as in Lemma 3, the last point of this section evidences the need to embed critical cycles in a larger, structural 'environment' to obtain G- and D-impact transitions. In other terms, $\mathcal{N}$ must have a critical cycle $\mathbf{C} = (\mathbf{V_C}, \mathbf{A_C})$ as well as nodes $i \in \mathbf{V} \setminus \mathbf{V_C} \neq \emptyset$ outside of it if the addition of synchronism is to significantly impact on its behaviour and change its asymptotics.

Lemma 4. *Let $\mathcal{N}$ be a* BAN *with no normal transitions of size smaller than its size n. Then, any transition $x \Longrightarrow y$ either has no impact on the asymptotics of $\mathcal{N}$ or it has F-impact. In the latter case, y is a stable with an empty basin $\mathcal{B}_{\mathrm{a}}(y) = \emptyset$ in the* ATG *and all nodes of the structure $\mathbf{G}$ of $\mathcal{N}$ have a positive loop.*

Proof. Let $x \Longrightarrow y = \overline{x}^{\mathbf{V}}$ be a normal transition of $\mathcal{N}$. Since $\forall i \in \mathbf{V},\ \mathbf{V} \setminus \{\, i \,\} \subset \mathbf{U}(x) = \mathbf{V}$, Proposition 2 implies $x \longrightarrow\!\!\triangleright\!\!\triangleright \overline{y}^{\,i}$, $\forall i \in \mathbf{V}$. Thus, $\forall z \in \mathbb{B}^n,\ y \longrightarrow\!\!\triangleright z \Rightarrow x \longrightarrow\!\!\triangleright\!\!\triangleright z$. And either $\mathbf{U}(y) \neq \emptyset$ in which case $\mathcal{A}_{\mathrm{a}}(y) \subset \mathcal{A}_{\mathrm{a}}(x)$ (and $x \Longrightarrow y$ has no impact), or y is stable in which case, by Lemma 3, its basin is empty and all automata bear a positive loop. $\square$

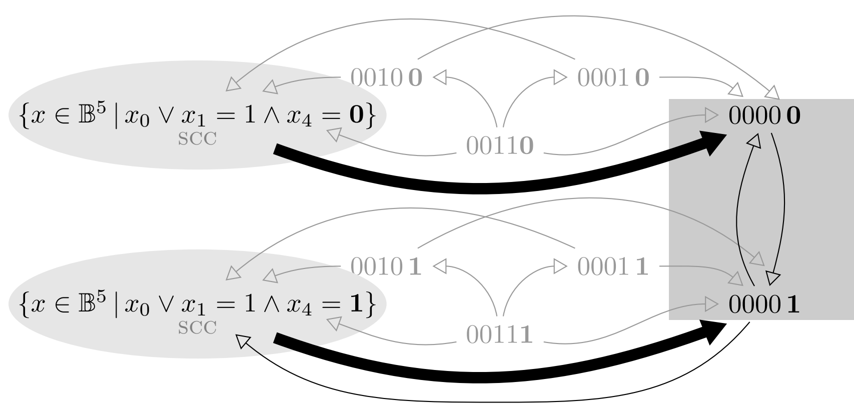
Example 1 (D- and G-impact). Let $\mathcal{N} = \{ f_i \,;\, i < 4 \}$ be the BAN of size 4 whose transition functions and signed structure are given below:

$$\forall x \in \mathbb{B}^n, \quad \begin{cases} \mathrm{h}_0(x) = x_2 \vee (x_0 \wedge \neg x_1) \\ \mathrm{h}_1(x) = x_3 \vee (\neg x_0 \wedge x_1) \\ \mathrm{h}_2(x) = \neg x_0 \wedge x_1 \\ \mathrm{h}_3(x) = x_0 \wedge \neg x_1 \end{cases}$$



The ETG of this BAN has two attractors: one unstable and one stable (configuration 0000). When $x_0 = x_1 = 1$ and $x_2 = x_3 = 0$, the simultaneous update of automata 0 and 1 has an effect that cannot be mimicked by a series of atomic updates (*cf.* Figure 3). If it could, strongly connected component $[1100]_\mathrm{a}$ would not be terminal in the ATG. From Propositions 1 and 2 we know that this is essentially due to the positive cycle of length 2 induced by automata 0 and 1.

Building on this example, we can derive an example of a BAN with a G-impact transition. Indeed, consider a fifth automaton $i = 4 \in \mathbf{V}$ s.t. $\mathrm{f}_4(x) = \neg(x_0 \vee x_1 \vee x_2 \vee x_3) \wedge \neg x_4$ and let $\mathrm{f}_0(x) = \mathrm{h}_0(x) \vee (x_4 \wedge \neg x_1 \wedge \neg x_3)$ and $\forall i \in \{1, 2, 3\}$, let $\mathrm{f}_i(x) = \mathrm{h}_i(x)$. It can be checked that $\mathrm{h}_0(x) \neq \mathrm{f}_0(x) \;\Rightarrow\; x = 00001$ and as a consequence, adding the same normal transition as before to the ATG of this new BAN yields the transition graph below:



Synchronism sensitivity

On the basis of the previous classification of the impact of synchronous transitions, we now take a more abstract point of view to propose a list of the different types of sensitivity that a BAN may have to (the addition of) synchronism. Naturally, we say that $\mathcal{N}$ has **no sensitivity to synchronism** if none of its normal synchronous transitions has any impact (*cf.* Point 1 in the previous section). We say that it has **F- and G-sensitivity to synchronism** when, respectively, it has normal transitions with F- and G-impact (*cf.* Point 2 and 3). When $\mathcal{N}$ has

normal transitions with D-impact (*cf.* Point 4), two cases may occur. Indeed, let $x \longrightarrow y$ be a normal transition of $\mathcal{N}$ that has D-impact. Then, there may be another D-impact normal transition $y' \longrightarrow x'$ such that $[x]_\mathrm{a} = [x']_\mathrm{a} \neq [y]_\mathrm{a} = [y']_\mathrm{a}$, *i.e.* x and x' on the one hand, and y and y' on the other belong to the same unstable attractors. In this case, the two normal transitions $x \longrightarrow y$ and $y' \longrightarrow x'$ cause attractors $[x]_\mathrm{a}$ and $[y]_\mathrm{a}$ of the ATG to **m**erge ($[x] = [y] = [x]_\mathrm{a} \uplus [y]_\mathrm{a}$). Hence, $\mathcal{N}$ is said to have **M-sensitivity to synchronism**. If there is no other normal transition connecting $[y]_\mathrm{a}$ to $[x]_\mathrm{a}$, then attractor $[x]_\mathrm{a}$ is effectively **d**estroyed by the addition of $x \longrightarrow y$ and $\mathcal{N}$ is said to have **D-sensitivity to synchronism**. This and the results presented above as well as, notably, the series of remarks made at the end of the previous section yield Proposition 3 below.

Proposition 3. *1. Sensitivity to synchronism requires the existence of a critical cycle, and thus of an positive cycle with an even length or a negative cycle with an odd length.*

2. G-, D- and M-sensitivity require the existence of a critical cycle of length strictly smaller than the BAN *size as well as of a negative cycle.*

3. Unless $\mathcal{N}$ has a Hamiltonian critical cycle and positive loops on all of its automata, to have F-sensitivity, $\mathcal{N}$ also needs to have a critical cycle of length strictly smaller than the BAN *size.*

Sensitivity to synchronism & non-monotony

Obviously, to be sensitive to synchronism, a BAN must involve at least two automata. It can be checked that there are no monotone BANs of size 2 that are D- or M-sensitive (we say **very sensitive**) to synchronism, but there are some non-monotone ones (*cf.* Figure 4).

However, interestingly, the monotone, D-sensitive BAN of Example 1 actually also involves non-monotone actions. Indeed, it only involves a few monotone individual interactions between four automata but these are architectured into a *widget* that can globally *mimic* a punctual non-monotone action in the right configuration and with the right synchronous update of automata states. More precisely but informally, in this widget, a non-monotone action is *structurally* split into two parts. These two parts consist in the two halves of a XOR: $(x_0\, x_1) \mapsto x_0 \wedge \neg x_1$ and $(x_0\, x_1) \mapsto \neg x_0 \wedge x_1$. They are encoded separately in the transition functions f_0 and f_1 of two different automata connected by what can be a critical cycle by Proposition 1. When the controls on these two parts are lifted (*i.e.* when $x_2 = x_3 = 0$ so that we do indeed have $\mathrm{f}_0(x) = x_0 \wedge \neg x_1$ and $\mathrm{f}_1(x) = \neg x_0 \wedge x_1$), the synchronous update of automata

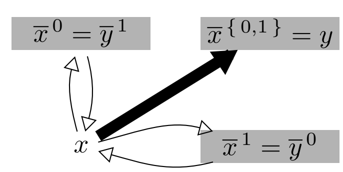


Fig. 4. ATG of a strongly connected BAN $\mathcal{N}$ of size 2 s.t. $\mathrm{f}_0, \mathrm{f}_1 \in \{x \mapsto x_0 \oplus x_1, x \mapsto \neg(x_0 \oplus x_1)\}$, augmented with normal transition $x \longrightarrow y$. $\mathcal{N}$ is D-sensitive to synchronism.

0 and 1 simultaneously applies f_0 and f_1. Instantly, this amounts to combining influences underwent by 0 and 1 by "simulating" a OR connector between their transition functions, thereby outputting the global action $\mathrm{f}_0(x) \vee \mathrm{f}_1(x)$. Precisely, this puts together the two halves of a XOR with a $\vee$ and produces a global non-monotone action. Examining the widget of Example 1, one can notice that the automata that it involves have different roles. Roughly, automata 0 and 1 encode the non-monotone action mentioned above. The role of automata 2 and 3 is to make "use" of it and ensure the necessary unstable attractor. This attractor is made dependent on automata 0 and 1 by requiring $x_0 \vee x_1 = 1$. More precisely, the widget is designed so that the unstable attractor is characterised by this condition. In the ATG, if the condition becomes true, it remains true. Every configuration x such that $x_0 = x_1 = 0$ reaches the stable configuration.

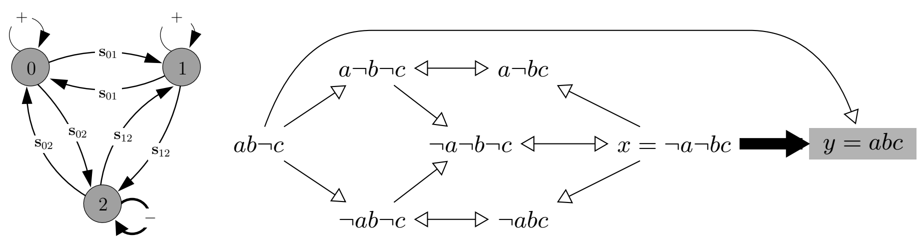


Fig. 5. Generic signed structure ($\forall i, j \in \mathbf{V}$, $\mathbf{s}_{ji} = \mathrm{sign}(j, i) = \mathrm{sign}(i, j)$) and modified ATG $\mathcal{T}_{\mathrm{a}}{}'$ of all monotone BANs of size 3 that are very sensitive (necessarily D-sensitive) to synchronism, *e.g.* $\mathcal{N} = \{\, \mathrm{f}_0 : x \mapsto x_2 \vee (x_0 \wedge \neg x_1),\ \mathrm{f}_1 : x \mapsto x_2 \vee (\neg x_0 \wedge x_1),\ \mathrm{f}_2 : x \mapsto \neg x_2 \wedge (x_0 \vee x_1) \,\}$. For all instances of these BANs, in the starting point x of the normal transition, $\mathrm{f}_2\big(\mathrm{f}_0(x)\,\mathrm{f}_1(x)\,x_2\big) \in \{\, x_0 \oplus x_1,\ \neg(x_0 \oplus x_1) \,\}$.

These remarks suggest that there is a tight relationship between significant sensitivity to synchronism and non-monotony[3]. Let us add that the smallest monotone BANs that are sensitive to synchronism have size 3. They are monotone encodings of the non-monotone sensitive BANs of size 2 (*cf.* Figure 4). This is proven by building around a normal transition $x \in \mathbb{B}^3 \longrightarrow y \in \mathbb{B}^3$ (that must satisfy $\mathrm{d}(x, y) = 2 < 3$ by Lemma 4) and substantially exploiting Lemmas 1 and 2 that hold with the hypothesis of monotony. From this we derive in particular that all such BANs have an ATG and a signed structure of the form of those represented in Figure 5, and they have D-sensitivity.

Conclusion and perspectives

Intuitively, for monotone BANs, frustrations and thus local instabilities are best maintained by synchronism. In particular, the parallel update schedule system-

[3] Notably, the example given by Robert in [16] to illustrate the "bursting of attractors" caused by parallelisation (which agrees with D-sensitivity) in a deterministic setting, can also be shown to involve non-monotony (it has size 3).

atically exploits all this momenta in each network configuration to perform all possible changes. Thus, it is known to have a tendency to induce what are sometimes considered as *artefact behaviours*, unstable and dependant on its strong constraint of synchronism. In [11] we conjectured and argued that the more "intricate" the network structure, that is, the more interconnected are its underlying cycles, the less chances do local instabilities have to be sustained. And this was observed under the parallel update schedule which is especially good at maintaining instabilities, so it seems that cycle intersections have a strong propensity to reduce the sensitivity of network behaviours to one of the characteristic effects of synchronism. In this paper, we investigated further in these lines, focusing on synchronism and its effect at a more minute level, that of elementary transitions. And we also related synchronism sensitivity to structural properties, namely the existence of critical cycles of positive sign and even length or of negative sign and odd length embedded in a particular environment. Also, notably, we have provided an example to evidence that synchronism in itself may indeed impact significantly on the asymptotic behaviour of a network: not only can it modify transient behaviours and make attractors grow, it can also destroy unstable attractors. Contrary to some traditional intuitions, asymptotically, asynchronism does not necessarily guaranty a minimum of local instability. Synchronism too can filter instabilities in asymptotic behaviours. The disregard that synchronism has had in theoretical modelling fields supports the importance of this: the time flow mentioned in the introduction – accounted for by the way automata states are updated, in particular synchronously or not – is a determining parameter of networks behaviours. What is more, we have argued in the previous section that non-monotony (typically and minimally captured by a logical XOR connector) might be necessary to manifest significant sensitivity to synchronism. It seems that monotone influences can be decomposed and stretched out in time whereas non-monotone influences require specific inputs at a precise instant (otherwise they act as a monotone influences, so these are defining conditions). Thus, non-monotony relies on dynamics. And it can be hardwired in the network definition or it may be *mimicked* punctually by synchronism. In the latter case, 'time flow' assembles basic, hardwired operators in a way that is otherwise impossible.

We have proven that sensitivity to synchronism requires strong structural properties. It deserves to be highlighted that this can also be interpreted in favour of asynchronous formalisations which advantageously yield less voluminous behaviour descriptions. Indeed, *most of the time*, synchronism only shortcuts asynchronous trajectories. However again, focusing on the asymptotics of a network behaviour, if complexity is more of an issue than exhaustivity, then deterministic update schedules such as the parallel update schedule also deserve to be investigated and studied for themselves. Thus, this work calls for further studies to better gauge the determining capacity of update schedules on the behaviours of networks. We have put forward (especially with Example 1) the existence of a "criticality" that involves updates whose effect is to decrease suddenly and non-reversibly the number of local instabilities. This establishes a new point of view on how networks work and this way, raises many new questions, *e.g. how*

must interactions between unstable automata be organised if their updates are to be consequential? and *how, generally, do local, punctual instabilities relate to the global (asymptotic) instability of a network?*.

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